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CALCULATION OF SPATIAL EQUILIBRIUM FORMS FOR THIN ELASTIC RODS BY THE SELF-EQUALIZING DISCREPANCY METHOD

G. V. Ivanov and O. N. Ivanova UDC 539.3

Algorithms for finding the planar equilibrium forms of thin elastic rods by the self-equalizing discrepancy method were presented in [1]. Below we will present algorithms for finding spatial equilibrium forms by the same method.

1. Basic Equations. To determine the position of the rod points, together with a fixed coordinate system x_1 with defining vectors e_i ($i = 1, 2, 3$) (Fig. 1) it will be convenient to use the arc length of the bar axis ($s \in [0, l]$) and coordinates x_1', x_2' with origin on the rod axis and defining vectors e_1', e_2', directed along the main axes of the bar cross sections. The orientation of e_i', $i = 1, 2, 3$ ($e_3' = e_1' \times e_2'$) is specified by the Euler angles ϑ, ψ, φ (Fig. 2).

The equilibrium equation can be written in the form [2, 3]

$$\frac{d\mathbf{F}}{ds} + \mathbf{f} = 0, \quad \frac{d\mathbf{M}}{ds} + \mathbf{e}_3' \times \mathbf{F} = 0, \tag{1.1}$$

where **F**, **M** are force vectors of the moments in the rod cross sections, **f** is the load surface vector,

$$\mathbf{e}_3' = \sin\vartheta\,(\sin\psi \cdot \mathbf{e}_1 - \cos\psi \cdot \mathbf{e}_2) + \cos\vartheta \cdot \mathbf{e}_3.$$

Below we will consider a class of problems in which the vector **F** is independent of ϑ, ψ, φ and is completely defined by the first expression of Eq. (1.1) and the boundary conditions. Below we will consider the function $\mathbf{F}=\mathbf{F}(s)$ to be known.

We assume [2, 3] that

$$\mathbf{M} = M_k \mathbf{e}_k', \quad M_1 = Ap, \quad M_2 = Bq, \quad M_3 = C\tau. \tag{1.2}$$

Here A, B, C are the rod rigidities relative to inflection and torsion, p, q are curvatures; τ is the torsion of the rod axis:

$$\begin{aligned} p &= \frac{d\psi}{ds}\sin\vartheta \cdot \sin\varphi + \frac{d\vartheta}{ds}\cos\varphi, \\ q &= \frac{d\psi}{ds}\sin\vartheta \cdot \cos\varphi - \frac{d\vartheta}{ds}\sin\varphi, \\ \tau &= \frac{d\psi}{ds}\cos\vartheta + \frac{d\varphi}{ds}. \end{aligned} \tag{1.3}$$

We will limit or treatment to the class of problems with boundary conditions

$$\vartheta = \psi = \varphi = x_1 = x_2 = x_3 = 0 \quad \text{for } s = 0; \tag{1.4}$$

$$\vartheta = \vartheta_0, \ \psi = \psi_0, \ \varphi = \varphi_0 \quad \text{for } s = l \tag{1.5}$$

or

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$$\mathbf{M} = \mathbf{0}. \tag{1.6}$$

With the assumptions made the problem solution reduces to integration of the second expression of Eq. (1.1) and Eqs. (1.2), (1.3) given conditions (1.4), (1.5) or (1.6).

After calculation of the functions ϑ, ψ, φ the coordinates of the rod axis x_k are calculated with the expression

$$x_k = \int_0^s \mathbf{e}_3' \cdot \mathbf{e}_k ds.$$

2. Variation Formulation of the Problem. We denote by $\delta\gamma$ the variation of the rotation vector

$$\delta\gamma = \mathbf{e}_3\delta\psi + \mathbf{e}_3'\delta\varphi + \mathbf{n}\delta\vartheta,\ \mathbf{n} = \cos\psi \cdot \mathbf{e}_1 + \sin\psi \cdot \mathbf{e}_2.$$

It can easily be shown that the variation $\delta\gamma$ can be written in the form

$$\delta\gamma = \mathbf{e}_k' \cdot \delta\gamma_k,\ \delta\gamma_1 = \sin\vartheta \cdot \sin\varphi \cdot \delta\psi + \cos\varphi \cdot \delta\vartheta,$$
$$\delta\gamma_2 = \sin\vartheta \cdot \cos\varphi \cdot \delta\psi - \sin\varphi \cdot \delta\vartheta,$$
$$\delta\gamma_3 = \cos\vartheta \cdot \delta\psi + \delta\varphi\ ,$$

while the derivatives de_k'/ds are related to p, q, τ by the equations

$$\frac{d\mathbf{e}_k'}{ds} = \omega \times \mathbf{e}_k',\ \omega = p\mathbf{e}_1' + q\mathbf{e}_2' + \tau\mathbf{e}_3'.$$

Hence it follows that

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$$\frac{d}{ds}\delta\gamma = \mathbf{e}_1'\delta p + \mathbf{e}_2'\delta q + \mathbf{e}_3'\delta\tau. \tag{2.1}$$

Multiplying the second expression of Eq. (1.1) by $\delta\gamma$, integrating over s with consideration of Eq. (2.1), and using the equality

$$(\mathbf{e}_3' \times \mathbf{F})\,\delta\gamma = \mathbf{F}(\delta\gamma \times \mathbf{e}_3') = \mathbf{F} \cdot \delta\mathbf{e}_3' = \delta(\mathbf{F} \cdot \mathbf{e}_3'),$$

we find that the problem solution corresponds to an extremum of the functional

$$\Phi = \frac{1}{2}\int_0^l (Ap^2 + Bq^2 + C\tau^2 - 2\mathbf{F} \cdot \mathbf{e}_3')ds \tag{2.2}$$

in the class of functions ϑ, ψ, φ which satisfies conditions (1.4), and if the boundary conditions at $s = l$ are formulated in the form of Eq. (1.5), then those conditions as well.

Below we will present an algorithm for construction of a sequence of approximations with monotonically decreasing values of functional (2.2), based on alternating variation of only one of the functions φ, ψ, φ.

3. Approximation of the Energy Functional by Quadratic Functionals. Let $\bar{\vartheta}(s)$, $\bar{\psi}(s)$, $\bar{\varphi}(s)$ be some functions equal to zero at $s = 0$ and satisfying conditions (1.5) if the latter are specific. In the proposed algorithm we construct functions $\bar{\vartheta}(s)$, $\bar{\psi}(s)$,$\bar{\varphi}(s)$ equal to zero at $s = 0$, and in the case of (1.5) also at $s = l$, for which

$$\Phi(\bar{\vartheta} + \dot{\vartheta}, \bar{\psi} + \dot{\psi}, \bar{\varphi} + \dot{\varphi}) \leq \Phi(\bar{\vartheta}, \bar{\psi}, \bar{\varphi}), \tag{3.1}$$

with the equality in Eq. (3.1) being satisfied when $\dot{\vartheta}(s)$, $\dot{\psi}(s)$, $\dot{\varphi}(s)$ are identically equal to zero, which is possible only when $\bar{\vartheta}$, $\bar{\psi}$, $\bar{\varphi}$ satisfy the conditions of the extremum of the functional Φ.

Construction of the functions $\dot{\vartheta}(s)$, $\dot{\psi}(s)$, $\dot{\varphi}(s)$ involves three stages. In each of these one of the functions is defined in turn.

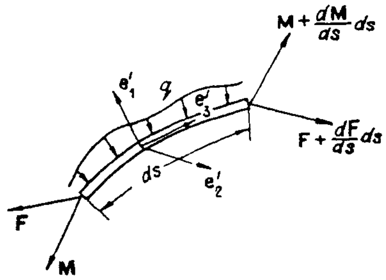

Fig. 1

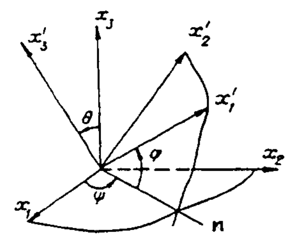

Fig. 2

In the first stage we find the function $\dot{\vartheta}(s)$, satisfying the inequality

$$\Phi(\tilde{\vartheta} + \dot{\vartheta}, \tilde{\psi}, \tilde{\varphi}) \leq \Phi(\tilde{\vartheta}, \tilde{\psi}, \tilde{\varphi}). \tag{3.2}$$

The functional $\Phi(\tilde{\vartheta} + \dot{\vartheta}, \tilde{\psi}, \tilde{\varphi})$ is approximated by a functional Φ_0 quadratic in $\dot{\vartheta}$

$$\Phi_\vartheta = \frac{1}{2}\int_0^l [A(\tilde{p} + \dot{p})^2 + B(\tilde{q} + \dot{q})^2 + C(\tilde{\tau} + \dot{\tau})^2 + \tilde{T}\dot{\vartheta}^2 + 2(\tilde{Q}\dot{\vartheta} - \tilde{T})]ds, \tag{3.3}$$

where $\tilde{p}$, $\tilde{q}$, $\tilde{\tau}$ are curvatures and torsion corresponding to functions $\tilde{\vartheta}$, $\tilde{\psi}$, $\tilde{\varphi}$;

$$\dot{p} = \dot{\vartheta}\frac{d\tilde{\psi}}{ds}\cos\tilde{\vartheta}\cdot\sin\tilde{\varphi} + \frac{d\dot{\vartheta}}{ds}\cos\tilde{\varphi};$$

$$\dot{q} = \dot{\vartheta}\frac{d\tilde{\psi}}{ds}\cos\tilde{\vartheta}\cdot\cos\tilde{\varphi} - \frac{d\dot{\vartheta}}{ds}\sin\tilde{\varphi};$$

$$\dot{\tau} = -\dot{\vartheta}\frac{d\tilde{\psi}}{ds}\sin\tilde{\vartheta};$$

$$\tilde{T} = \tilde{H}\sin\tilde{\vartheta} + F_3\cos\tilde{\vartheta};\ \tilde{Q} = -\tilde{H}\cos\tilde{\vartheta} + F_3\sin\tilde{\vartheta};$$

$$\tilde{H} = F_1\sin\tilde{\psi} - F_2\cos\tilde{\psi}$$

F_k are the components of the vector **F** in the fixed coordinate system x_k ($k = 1, 2, 3$). The approximation is performed by the following rule; p, q, τ are replaced by the linear portions of expansions in a series in $\dot{\vartheta}$, and $\mathbf{F}\cdot\mathbf{e}_3'$, by the quadratic portion of an expansion in a series in $\dot{\vartheta}$.

For sufficiently small values of the function $\dot{\vartheta}(s)$ from the inequality

$$\Phi_\vartheta(\tilde{\vartheta} + \dot{\vartheta}, \tilde{\psi}, \tilde{\varphi}) \leq \Phi_\vartheta(\tilde{\vartheta}, \tilde{\psi}, \tilde{\varphi}) \tag{3.4}$$

we have inequality (3.2). Therefore the problem of reducing functional (2.2) by selection of $\dot{\vartheta}(s)$ can be regarded as the problem of defining a function $\dot{\vartheta}(s)$ which satisfies Eq. (3.4) and the condition

$$\max_s|\dot{\vartheta}(s)| \leq \alpha_1$$

(where α_1 is a constant which can be adjusted during definition of $\dot{\vartheta}(s)$ such that Eq. (3.2) follows from Eq. (3.4).

In the second stage the functional $\Phi(\tilde{\vartheta}, \tilde{\psi} + \dot{\psi}, \tilde{\varphi})$ ($\tilde{\vartheta}' = \tilde{\vartheta} + \dot{\vartheta}$) is approximated by a functional Φ_ψ quadratic in $\dot{\psi}$:

$$\begin{aligned}\Phi_\psi = \frac{1}{2}\int_0^l &[A(\tilde{p}' + \dot{p}')^2 + B(\tilde{q}' + \dot{q}')^2 + C(\tilde{\tau}' + \dot{\tau}')^2 \\ &+ \dot{\psi}^2\tilde{H}\sin\tilde{\vartheta}' - 2(\tilde{T}' + \dot{\psi}\tilde{N}'\sin\tilde{\vartheta}')]ds.\end{aligned} \tag{3.5}$$

Here $\tilde{p}'$, $\tilde{q}'$, $\tilde{\tau}'$ are curvatures and torsion corresponding to the functions $\tilde{\vartheta}$, $\tilde{\psi}$, $\tilde{\varphi}$;

TABLE 1

γ	b/a
0,140	1
0,228	2
0,281	4
0,307	8
0,312	10

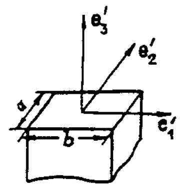

Fig. 3

$$\dot{p}' = \frac{d\dot{\psi}}{ds}\sin\tilde{\vartheta}' \cdot \sin\tilde{\varphi};\ \dot{q}' = \frac{d\dot{\psi}}{ds}\sin\tilde{\vartheta}' \cdot \cos\tilde{\varphi};\qquad \dot{\tau}' = \frac{d\dot{\psi}}{ds}\cos\tilde{\vartheta}';$$
$$\tilde{T}' = \tilde{H}\sin\tilde{\vartheta}' + F_3\cos\tilde{\vartheta}';\ \tilde{N}' = F_1\cos\tilde{\psi} + F_2\sin\tilde{\psi}.$$

The approximation is performed by the same rule by which functional (3.3) was constructed.

The function $\dot{\psi}(s)$ is defined by the conditions

$$\Phi_\psi(\tilde{\vartheta}', \tilde{\psi} + \dot{\psi}, \tilde{\varphi}) \leq \Phi_\psi(\tilde{\vartheta}', \tilde{\psi}, \tilde{\varphi}),\ \max_s|\dot{\psi}(s)| \leq \alpha_2,$$

where α_2 is a constant selected to achieve the inequality

$$\Phi(\tilde{\vartheta}', \tilde{\psi} + \dot{\psi}, \tilde{\varphi}) \leq \Phi(\tilde{\vartheta}', \tilde{\psi}, \tilde{\varphi}).$$

In the third stage the functional $\Phi(\tilde{\vartheta}, \tilde{\psi}', \tilde{\varphi} + \dot{\varphi})$ $(\tilde{\psi}' = \tilde{\psi} + \dot{\psi})$ is approximated by a functional Φ_φ quadratic in $\dot{\varphi}$:

$$\Phi_\varphi = \frac{1}{2}\int_0^l [A(\tilde{p}'' + \dot{p}'')^2 + B(\tilde{q}'' + \dot{q}'')^2 + C(\tilde{\tau}'' + \dot{\tau}'')^2 - 2\tilde{T}'']ds. \tag{3.6}$$

Here $\tilde{p}''$, $\tilde{q}''$, $\tilde{\tau}''$ is approximated by a functional $\tilde{\vartheta}$, $\tilde{\psi}'$, $\tilde{\varphi}$;

$$\dot{p}'' = \left(\frac{d\tilde{\psi}'}{ds}\sin\tilde{\vartheta}' \cdot \cos\tilde{\varphi} - \frac{d\tilde{\vartheta}'}{ds}\sin\tilde{\varphi}\right)\dot{\varphi};$$
$$\dot{q}'' = -\left(\frac{d\tilde{\psi}'}{ds}\sin\tilde{\vartheta}' \cdot \sin\tilde{\varphi} + \frac{d\tilde{\vartheta}'}{ds}\cos\tilde{\varphi}\right)\dot{\varphi};$$
$$\dot{\tau}'' = \frac{d\dot{\varphi}}{ds},\ \tilde{T}'' = \sin\tilde{\vartheta}'(F_1\sin\tilde{\psi}' - F_2\cos\tilde{\psi}') + F_3\cos\tilde{\vartheta}'.$$

The approximation is performed by the same rule used for Eqs. (3.3), (3.5). The function $\dot{\varphi}(s)$ is defined by the condition

$$\Phi_\varphi(\tilde{\vartheta}', \tilde{\psi}', \tilde{\varphi} + \dot{\varphi}) \leq \Phi_\varphi(\tilde{\vartheta}', \tilde{\psi}', \tilde{\varphi}),\ \max_s|\dot{\varphi}(s)| \leq \alpha_3,$$

where α_3 is a constant selected to achieve the inequality

$$\Phi(\tilde{\vartheta}', \tilde{\psi}', \tilde{\varphi} + \dot{\varphi}) \leq \Phi(\tilde{\vartheta}', \tilde{\psi}', \tilde{\varphi}).$$

TABLE 2

N	Φ'	$x_1(1)$	$x_2(1)$	$x_3(1)$
40	$-1{,}078\cdot10^{-3}$	0,0707	−0,2382	0,9570
80	$-1{,}078\cdot10^{-3}$	0,0711	−0,2390	0,9566
100	$-1{,}078\cdot10^{-3}$	0,0711	−0,2392	0,9566

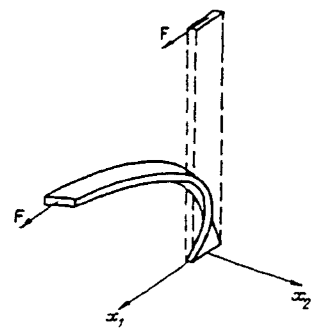

Fig. 4

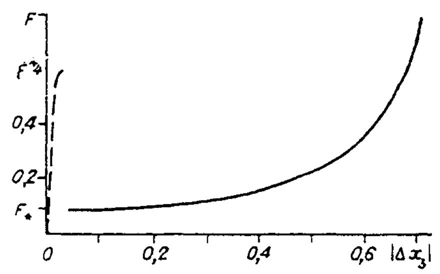

Fig. 5

4. Difference Approximation. The bar axis is divided into N elements by the points $s_i = (i-1)/h$ ($i = 1, 2, \ldots, N+1$, $h = l/N$). These elements are enumerated by numbers $i + 1/2$ ($i = 1, 2, \ldots, N$) and quantities corresponding to the element $i + 1/2$ bear the index $i + 1/2$. We set

$$\vartheta_{i+1/2} = \frac{1}{2}(\vartheta_i + \vartheta_{i+1}),\ \left(\frac{d\vartheta}{ds}\right)_{i+1/2} = \frac{1}{h}(\vartheta_{i+1} - \vartheta_i)$$

(where ϑ_i is the value of ϑ at the points). Similarly, in terms of ψ_i, φ_i we define the quantities

$$\psi_{i+1/2},\ \varphi_{i+1/2},\ \left(\frac{d\psi}{ds}\right)_{i+1/2},\ \left(\frac{d\varphi}{ds}\right)_{i+1/2}.$$

at the points. The cosines and sines in the element $i + 1/2$ are calculated with the expressions

$$(\cos\vartheta)_{i+1/2} = \cos\vartheta_{i+1/2},\ (\sin\varphi)_{i+1/2} = \sin\varphi_{i+1/2}.$$

Functional (2.2) is replaced by the difference expression

$$\Phi' = \frac{1}{2}h\sum_{i=1}^{N}\left[Ap^2_{i+1/2} + Bq^2_{i+1/2} + C\tau^2_{i+1/2} - 2\mathbf{F}_{i+1/2}(\mathbf{e}'_3)_{i+1/2}\right], \tag{4.1}$$

where

$$p_{i+1/2} = \frac{1}{h}\left[(\psi_{i+1} - \psi_i)\sin\vartheta_{i+1/2}\cdot\sin\varphi_{i+1/2} + (\vartheta_{i+1} - \vartheta_i)\cos\varphi_{i+1/2}\right],$$

and the quantities

$$q_{i+1/2},\ \tau_{i+1/2},\ \mathbf{F}_{i+1/2},\ (\mathbf{e}'_3)_{i+1/2}\,.$$

TABLE 3

Number of iterations	Φ'	$x_1(1)$	$x_2(1)$	$x_3(1)$
0	28,43	0	−0,740	−0,110
30	19,92	$2{,}7\cdot10^{-2}$	−0,479	$6{,}3\cdot10^{-2}$
100	19,75	$1{,}9\cdot10^{-2}$	−0,345	$1{,}8\cdot10^{-2}$
120	19,75	$1{,}9\cdot10^{-2}$	−0,331	$2{,}5\cdot10^{-2}$

have a similar meaning. Functionals (3.3), (3.5), (3.6) are replaced by the corresponding difference functionals Φ_ϑ', Φ_ψ', Φ_φ'. The quadratic portions Δ_0, Δ_ψ of the functionals Φ_ϑ', Φ_ψ' have the following values:

$$\Delta_\vartheta = \frac{1}{2}h\sum_{i=1}^{N}[A\dot{p}^2_{i+1/2} + B\dot{q}^2_{i+1/2} + C\dot{\tau}^2_{i+1/2} + \bar{T}_{i+1/2}\,\dot{\vartheta}^2_{i+1/2}]$$

$$\geqslant \frac{1}{2}h\sum_{i=1}^{N}\left[D_*\left(\frac{d\dot{\vartheta}}{ds}\right)^2_{i+1/2} - T_*\dot{\vartheta}^2_{i+1/2}\right],$$

$$\Delta_\psi = \frac{1}{2}h\sum_{i=1}^{N}\left[A(\dot{p}'_{i+1/2})^2 + B(\dot{q}'_{i+1/2})^2 + C(\dot{\tau}'_{i+1/2})^2 + \bar{H}'_{i+1/2}\sin\bar{\vartheta}'_{i+1/2}\cdot\dot{\psi}^2_{i+1/2}\right]$$

$$\geqslant \frac{1}{2}h\sum_{i=1}^{N}\left[D_*\left(\frac{d\dot{\psi}}{ds}\right)^2_{i+1/2} - T_*\dot{\psi}^2_{i+1/2}\right],$$

$$D_* = \min(A, B, C),\ T_* = \max_i \sum_{k=1}^{3}|(F_k)_{i+1/2}|.$$

The portion of the functional Φ_φ' quadratic in $\dot{\varphi}_i$ is a passively defined form. From this it follows that to minimize the functionals Φ_ϑ', Φ_ψ', Φ_φ' given the condition

$$h \leqslant \frac{D_*}{T_* l}$$

we can use the self-equalizing discrepancy method of [1], thus generating values of $\dot{\vartheta}_i$, $\dot{\psi}_i$, $\dot{\varphi}_i$ ($i = 1, 2, \ldots, N + 1$) for which

$$\Phi'(\bar{\vartheta}_i + \dot{\vartheta}_i, \bar{\psi}_i + \dot{\psi}_i, \bar{\varphi}_i + \dot{\varphi}_i) \leqslant \Phi'(\bar{\vartheta}_i, \bar{\psi}_i, \bar{\varphi}_i),$$

with equality occurring only in the case where $\bar{\vartheta}_i$, $\bar{\psi}_i$, $\bar{\varphi}_i$ satisfy the conditions of an extremum of the functional Φ' and the quantities $\dot{\vartheta}_i$, $\dot{\psi}_i$, $\dot{\varphi}_i$ ($i = 1, 2, \ldots, N + 1$) are thus equal to zero.

Thus, using the self-equalizing discrepancy method of [1], we can construct a sequence of approximate solutions with monotonically decreasing values of the functional Φ'. The sequence will converge, since the functional is bounded below:

$$\Phi' \geqslant -T_* l.$$

5. Examples of Equilibrium Form Calculation. We will consider rods of rectangular cross section (Fig. 3). In this case

$$A = Ea^3b/12,\ B = Eab^3/12,\ C = \gamma Eba^3/2(1 + \nu).$$

The γ values are taken equal to those indicated in [4]. Some are presented in Table 1.

Below we will use dimensionless quantities: s, x_k, ratios of arc length and coordinate to bar length l; p, q, τ, curvatures and torsion multiplied by l; A, B, C, ratios of rigidities to B; F_k, M_k, components of force and moment vectors multiplied by l^2/B and l/B respectively; Φ', functional (4.1) multiplied by l/B.

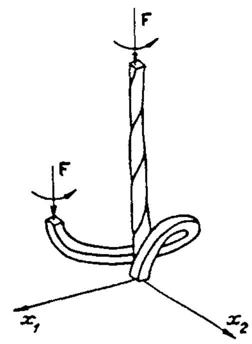

Fig. 6

In one of the examples we consider equilibrium states for inflection of intensity $\mathbf{F} = F\mathbf{e}_1$ of a bar with side length ratiovalue $b/a = 8$ (Fig. 4). It was assumed that the bar end $s = 1$ could rotate freely. Figure 5 shows the dependence of F on absolute $|\Delta x_3|$ of the vertical displacement of the bar end $s = 1$ ($|\Delta x_3| = 1 - x_3$, where x_3 is the coordinate of the bar end $s = 1$ in the deformed state). The dashed line corresponds to the planar equilibrium state, the solid, to the three-dimensional, $F^* = 0.6$ is the upper critical load [5], $F_* = 0.08$, the lower critical load.

Table 2 presents values of the functional Φ' and bar end coordinate $s = 1$, corresponding to $F = 0.08$ and division of the bar axis into 40, 80, and 100 elements. These values practically coincide. The criterion used for halting the calculation process was constancy to three significant figures in the values of the energy functional and the bar end coordinate. The number of iterations required to obtain the equilibrium forms was practically independent of number of elements, into which the bar axis was divided, being approximately equal to 50.

We also considered equilibrium states for compression of a twisted bar of square cross section with boundary conditions

$$\vartheta(0) = \psi(0) = \varphi(0) = \vartheta(1) = \psi(1) = 0,$$
$$\varphi(1) = 2\pi, \ \mathbf{F}(1) = -9\mathbf{e}_3.$$

Together with the equilibrium state in which the bar axis is rectilinear ($\vartheta = \psi = 0$, $\varphi = 2\pi s$), an equilibrium state of the form shown in Fig. 6 is possible. To calculate this state the initial approximation used was

$$\vartheta = 10(s - s^2), \ \psi = 0, \ \varphi = 2\pi s.$$

Table 3 shows the decrease in the energy functional and change in rod end coordinate during the iteration process for $N = 40$. In the example considered the functional Φ' decreases monotonically with no limitations on the values of variations of the unknown functions calculated by the self-equalizing discrepancy method [1].

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